Quasiseparation of variables in the Schrödinger equation with a magnetic field

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Abstract

We consider a two-dimensional integrable Hamiltonian system with a vector and scalar potential in quantum mechanics. Contrary to the case of a pure scalar potential, the existence of a second order integral of motion does not guarantee the separation of variables in the Schrödinger equation. We introduce the concept of "quasiseparation of variables" and show that in many cases it allows us to reduce the calculation of the energy spectrum and wave functions to linear algebra.

1 Introduction

A systematic search for integrable classical Hamiltonian systems in magnetic fields was started quite some time ago [3, 16, 17]. A Hamiltonian containing scalar and vector potentials was introduced in a two-dimensional Euclidean space and additional integrals of motion were constructed as linear or quadratic polynomials in the momenta. The same problem in quantum mechanics was investigated quite recently [2]. Some new features have emerged in this study of vector potentials that distinguish this case from that of purely scalar ones.

1. The existence of second order integrals of motion does not imply the separation of variables in the Hamilton-Jacobi, or Schrödinger equation.

2. Hamiltonian systems with second order integrals of motion in classical and in quantum mechanics do not necessarily coincide [2].

3. In quantum mechanics an additional problem arises: it is necessary to choose a convenient gauge in which to write and solve the Schrödinger equation.

In the case of Hamiltonians with purely scalar potentials, quantum and classical integrable systems may also differ, but only if the integrals of motion are third, or higher order in the momenta [7, 8, 9, 10]. Moreover, third and higher order integrals of motion are not related to the separation of variables, at least in configuration space, in the case of purely scalar potentials either [9, 10].

Variable separation in the presence of magnetic fields has also been investigated [1, 22, 23] and turns out to be quite rare. Separable systems in this case only constitute a subset of the quadratically integrable ones.

Natural questions that arise, specially in the context of quantum mechanics, are the following. What does one do with integrals of motion that do not lead to the separation of variables? How do they help to integrate the Schrödinger equation and find the energy spectrum?

The purpose of this article is to provide at least a partial answer to these questions.

The problem is formulated mathematically in Section 2. Section 3 is devoted to systems with one first order operator X, commuting with the Hamiltonian H. In Section 4 the commuting operator X is assumed to be a second order operator of a specific ("cartesian") type. We introduce the concept of quasiseparation of variables and identify all cases when quasiseparation occurs. The Schrödinger equation is solved for a specific scalar and vector potential in Section 5. Section 6 is devoted to conclusions.

2 General setting

The quantum Hamiltonian that we are considering is

$$H = -\frac{\hbar^2}{2}(\partial_x^2 + \partial_y^2) - \frac{i\hbar}{2}(A\partial_x + \partial_x A + B\partial_y + \partial_y B) + V, \tag{2.1}$$

where A, B and V are functions of the coordinates x,y. The quantities of direct physical importance are the magnetic field Ω and effective potential W:

$$\Omega = A_y - B_x, \;\; W = V - \frac{1}{2}(A^2 + B^2). \tag{2.2}$$

These quantities are gauge invariant, i.e. unchanged by the transformation

$$\begin{aligned} V \to \tilde{V} &= V + (\mathbf{A}, \nabla\phi) + \frac{1}{2}(\nabla\phi)^2, \\ \mathbf{A} \to \tilde{\mathbf{A}} &= \mathbf{A} + \nabla\phi, \;\; \mathbf{A} = (A, B), \end{aligned} \tag{2.3}$$

where ϕ(x,y) is an arbitrary smooth function.

The classical equations of motion are

$$\ddot{x} = \Omega\dot{y} - W_x, \;\; \ddot{y} = -\Omega\dot{x} - W_y, \tag{2.4}$$

so all we need to know is Ω and W (the dots indicate time derivatives). In quantum mechanics we need the Hamiltonian (2.1). Hence, given Ω and W, we must still choose a gauge, i.e. the function ϕ in eq. (2.3), and calculate A, B and V.

The quantities Ω and W are obtained from the commutativity condition

$$[X, H] = 0, \tag{2.5}$$

where X is the integral of motion, i.e. either a first, or a second order linear differential operator. The integral X is obtained from the same condition.

The two equations to solve simultaneously, once a gauge is chosen, are

$$H\psi = E\psi, \;\; X\psi = \lambda\psi, \tag{2.6}$$

and the vector potential (i.e the gauge) should be chosen so as to simplify this pair.

3 First order integrals

A first order integral of motion will have the form [2]

$$X = \alpha(L_3 + yA - xB) + \beta(P_1 + A) + \gamma(P_2 + B) + m, \tag{3.1}$$

where L_3, P_1 and P_2 are the angular and linear momentum operators, i.e.

$$L_3 = -i\hbar(y\partial_x - x\partial_y), \;\; P_1 = -i\hbar\partial_x, \;\; P_2 = -i\hbar\partial_y, \tag{3.2}$$

α, β and γ are constants and the functions m(x,y), Ω and W satisfy

$$\begin{gathered} (\alpha x - \gamma)\Omega + m_x = 0, \;\; (\alpha y + \beta)\Omega + m_y = 0, \\ (\alpha y + \beta)W_x + (-\alpha x + \gamma)W_y = 0. \end{gathered} \tag{3.3}$$

Two inequivalent possibilities occur. For $\alpha \neq 0$ we can put $\alpha=1$ and translate β and γ into $\beta=\gamma=0$. For $\alpha=0$, $\beta^2+\gamma^2 \neq 0$ we can rotate β into $\beta=0$ and normalize γ to $\gamma=1$. Let us now look at the two cases separately and solve the Schrödinger equation in each case.

a) $\alpha=0$, $\beta=0$, $\gamma=1$

We have [2]

$$m = m(x), \;\; W = W(x), \;\; \Omega = \Omega(x) = \dot{m}(x). \tag{3.4}$$

The integral of motion (3.1) in this case reduces to

$$X = P_2 + m(x) + B(x,y). \tag{3.5}$$

A convenient choice of gauge is

$$B = -m(x), \;\; A = 0, \tag{3.6}$$

and from (2.2) we obtain

$$\Omega = \dot{m}(x), \;\; V = W(x) + \frac{1}{2}m(x)^2. \tag{3.7}$$

The system (2.6) reduces to

$$\begin{gathered}\{-\frac{\hbar^2}{2}(\partial_x^2+\partial_y^2) + i\hbar m(x)\partial_y + W(x) + \frac{1}{2}m(x)^2\}\psi = E\psi, \\ i\hbar\partial_y\psi = k\psi.\end{gathered} \tag{3.8}$$

With this choice of gauge we have the separation of variables, i.e.

$$\psi(x,y) = f_{Ek}(x)e^{-\frac{ik}{\hbar}y}, \tag{3.9}$$

$$-\frac{\hbar^2}{2}\frac{d^2f}{dx^2} + [\frac{k^2}{2} + km + W + \frac{1}{2}m^2 - E]f = 0. \tag{3.10}$$

Equation (3.10) is exactly solvable for many choices of m(x) (i.e. for many choices of the magnetic field Ω) and the scalar potential V(x). For instance, if we have

$$m = \omega^2x^2, \;\; \Omega = 2\omega^2 x, \;\; V(x) = 0, \tag{3.11}$$

eq. (3.10) has the solution

$$f = e^{-\frac{\tau^2x^2}{2}}H_n(\tau x), \;\; \tau = \sqrt{\frac{\omega}{\hbar}}(2k)^{1/4}, \;\; E = \frac{k^2}{2} + (2n+1)\frac{\hbar\omega}{2}\sqrt{2k}, \tag{3.12}$$

where $H_n(\xi)$ is a Hermite polynomial. In this case f(x) is regular and square integrable, however the solution (3.9) involves a plane wave in y.

b) α=1, β=γ=0

In this case the determining equations (3.3) imply [2]

$$m = m(r), \;\; W = W(r), \;\; \Omega(r) = -\frac{\dot{m}}{r}, \tag{3.13}$$

where we are using polar coordinates

$$x = r cos\Theta, \;\; y = r sin\Theta. \tag{3.14}$$

The operator (3.1) in this case is

$$X = L_3 + yA - xB + m, \tag{3.15}$$

and a good choice of gauge is given by

$$yA - xB + m = 0, \tag{3.16}$$

leading to

$$A(r,\Theta) = -\frac{m(r)}{r} sin\Theta, \;\; B(r,\Theta) = \frac{m(r)}{r} cos\Theta, \tag{3.17}$$

$$V(r,\Theta) = W(r) + \frac{m^2}{2r^2}. \tag{3.18}$$

The solution of the system (2.6) can be written in separated forms as

$$\psi(r,\Theta) = e^{-iM\Theta}\frac{1}{\sqrt{r}}R_{E,M}(r), \tag{3.19}$$

with R(r) satisfying

$$-\frac{\hbar^2}{2}\frac{d^2R}{dr^2} + \{W(r) + \frac{1}{r^2}[\frac{\hbar^2 M^2}{2} + \frac{m(r)^2}{2} - \hbar m(r) M - \frac{\hbar^2}{8}]\}R = ER. \tag{3.20}$$

Again, this equation is exactly solvable in special cases, e.g.

$$m(r) = \frac{\alpha}{r}, \;\; W = -\frac{\alpha^2}{2r^4}. \tag{3.21}$$

The conclusion from this section is that first order integrability in a magnetic field leads to a separation of variables either in cartesian, or in polar coordinates. To make this happen, a proper choice of gauge is crucial. Indeed, in a previous article [2] a different choice of gauge was made, leading to R-separation, rather than ordinary separation. The separated ordinary differential equations (3.10) and (3.20) both have the form of one-dimensional Schrödinger equations. The magnetic field and the effective potential combine together into an x-dependent, or respectively r-dependent one-dimensional "potential".

4 Second order Cartesian integrability and the quasiseparation of variables

Let us now consider a Hamiltonian of the form (2.1), admitting one second order integral of motion X. It has been shown that this operator X (or function in classical mechanics) can be transformed into one of four standard forms [2,3]. They were called Cartesian, polar, parabolic and elliptic, because in the absence of a magnetic field, their existence leads to the separation of variables in Cartesian, polar, parabolic or elliptic coordinates, respectively [6, 20]. The direct relation to the separation of variables in the Schrödinger, or Hamilton-Jacobi equation does not hold in the presence of a magnetic field, but we keep the terminology. In this article we restrict ourselves to the Cartesian case.

The "Cartesian" integral of motion has the form [2, 3]

$$X = -\frac{\hbar^2}{2}\partial_x^2 - i\hbar[(A+k_1)\partial_x + k_2\partial_y] - \frac{i\hbar}{2}(A_x + k_{1x} + k_{2y}) + \frac{1}{2}A^2 \\ +m + k_1 A + k_2 B. \tag{4.1}$$

All functions involved in the Hamiltonian (2.1) and the integral (4.1) can be expressed in terms of two functions of one variable each, $f = f(x)$ and $g = g(y)$, satisfying

$$\ddot{f} = \alpha f^2 + \beta f + \gamma, \;\; g^{''} = -\alpha g^2 + \delta g + \xi, \tag{4.2}$$

where α, β, γ, δ and ξ are real constants. We shall also use the first integrals of eq. (4.2), namely

$$\dot{f}^2 = P_3(f), \qquad g^{'2} = Q_3(g) \tag{4.3}$$
$$P_3(f) = \frac{2}{3}\alpha f^3 + \beta f^2 + 2\gamma f + \sigma_1, \qquad Q_3(g) = -\frac{2}{3}\alpha g^3 + \delta g^2 + 2\xi g + \sigma_2$$

where σ_1 and σ_2 are further real constants. The dots and primes are x and y derivatives, respectively.

Eq. (4.2) and (4.3) can be solved in terms of elliptic functions, or their degeneracies, if the polynomials $P_3(f)$, or $Q_3(g)$ have multiple roots. In terms of $f(x)$ and $g(y)$ we have

$$\begin{gathered} \Omega = \ddot{f}(x) + g^{''}(y), \\ W = -\frac{\alpha}{3}(f-g)^3 - \frac{\beta+\delta}{2}(f-g)^2 + (\xi - \gamma + \mu)(f-g), \\ k_1 = -g^{'}(y), \;\; k_2 = -\dot{f}(x), \\ m = -\frac{\alpha}{3}(g^3 + 2f^3 - 3gf^2) + \beta(fg - f^2) - \frac{\delta}{2}(f^2 - g^2) \\ -\gamma(2f - g) + \mu f + \xi g. \end{gathered} \tag{4.4}$$

where μ, figuring in the effective potential W and in m is an additional constant. The results (4.2) and (4.4) were obtained in classical mechanics [3], but the classical and quantum results coincide in the case of Cartesian integrability [2].

The vector potential (A,B) is yet to be chosen, but must satisfy

$$A_y - B_x = \Omega = \ddot{f}(x) + g^{''}(y). \tag{4.5}$$

We shall in this section assume $\ddot{f}^2 + g^{''2} \neq 0$.

Thus, we have a Hamiltonian H and first integral X satisfying (2.1) and (4.1), respectively, expressed in terms of quantities satisfying eq. (4.4). In general, variables do not separate in eq. (2.6) in any system of coordinates.

Let us introduce the concept of "quasiseparation of variables". We take a linear combination of the two equations (2.6), namely

$$\{(H - E) + \phi(x, y)(X - \lambda)\}\Psi = 0 \tag{4.6}$$

where $\phi(x, y)$ is a function to be determined. We wish to choose the function $\phi(x, y)$ in such a manner that eq. (4.6) allows separation in Cartesian coordinates. The solutions of eq. (4.6) will then have the separated form

$$\Psi_{E\lambda}(x, y) = v_{E\lambda}(x) w_{E\lambda}(y). \tag{4.7}$$

The solution of the Schrödinger equation will be a superposition of separated solutions:

$$\Psi_E(x, y) = \int A_{E\lambda} v_{E\lambda}(x) w_{E\lambda}(y) d\mu(\lambda) \tag{4.8}$$

where $\mu(\lambda)$ is some measure to be chosen and $A_{E\lambda}$ is independent of x and y. For bound states the integral in eq. (4.8) will reduce to a sum.

On a more formal level we introduce the following definition.

Definition 1. *The commuting pair of operators $\{H, X\}$ allows the* **quasiseparation of variables** *in the system (2.6) if there exists a function $\phi(x, y)$ such that eq. (4.6) allows the separation of variables in the sense of eq. (4.7).*

In this article we are considering the case when (x,y) are Cartesian coordinates in a plane, but the concept of quasiseparation is easily generalized to other coordinates and other spaces.

Substituting (4.7) into (4.6) we obtain the equation

$$-\frac{\hbar^2}{2} v w^{''} + \phi_1 \ddot{v} w + \phi_2 \dot{v} w + \phi_3 v w^{'} + \phi_4 v w = 0 \tag{4.9}$$

with

$$\phi_1 = -\frac{\hbar^2}{2}(1 + \phi), \quad \phi_2 = -i\hbar A(1 + \phi) + i\hbar\phi g^{'}, \quad \phi_3 = -i\hbar(B - \phi\dot{f}), \tag{4.10}$$

$$\phi_4 = W + \frac{1}{2}A^2(1+\phi) + \frac{1}{2}B^2 + \phi(m - g^{'}A - \dot{f}B) \\ -\frac{i\hbar}{2}((1+\phi)A_x + B_y) - E - \lambda\phi. \tag{4.11}$$

The necessary and sufficient condition for variables to separate in eq. (4.9) is that we have

$$\begin{aligned} \frac{\phi_1}{\phi_4} &= \frac{V_1(x)}{\tilde{V}(x)+\tilde{W}(y)} & \frac{\phi_2}{\phi_4} &= \frac{V_2(x)}{\tilde{V}(x)+\tilde{W}(y)} \\ \frac{-\hbar^2}{2\phi_4} &= \frac{W_1(y)}{\tilde{V}(x)+\tilde{W}(y)} & \frac{\phi_3}{\phi_4} &= \frac{W_2(y)}{\tilde{V}(x)+\tilde{W}(y)} \end{aligned} \tag{4.12}$$

where V_i, W_i, $\tilde{V}$ and $\tilde{W}$ are some functions.

Let us consider the choice $\phi(x,y) = -1$ separately.

4.1 $\phi(x,y) = -1$

With this particular choice, eq. (4.9) simplifies, $V_2(x)$ reduces to a constant, and we find that

$$B(x,y) = -\dot{f} + \frac{W_2(y)}{V_2}g^{'}, \qquad \textit{and} \qquad W_1(y) = \frac{\hbar}{2i}\frac{V_2}{g^{'}}. \tag{4.13}$$

Condition (4.5) becomes

$$A(x,y) = g^{'} + \tau(x), \tag{4.14}$$

where $\tau(x)$ is arbitrary, and eqs. (4.12) reduce to

$$\frac{g^{'}}{\phi_4} = -\frac{V_2}{i\hbar(\tilde{V}(x)+\tilde{W}(y))}. \tag{4.15}$$

Thus

$$\frac{\partial^2}{\partial x \partial y}(\frac{\phi_4}{g^{'}}) = 0 \tag{4.16}$$

is a necessary and sufficient condition for quasiseparation. A straightforward calculation shows that this condition only depends on g(y) and is equivalent to

$$g^{''2} - g^{'}g^{'''} = 0. \tag{4.17}$$

The general solution of eq. (4.17) is $g(y) = C_1 e^{C_2 y} + C_3$ and a two parameter class of particular solutions is $g(y) = g_1 y + g_0$. It follows that $\phi(x,y) = -1$ only allows quasiseparation in the cases $\alpha = \delta = \xi = 0$ and $\alpha = 0$, $\delta > 0$ (i.e. $g(y) = g_1 e^{\sqrt{\delta}y} + g_2 e^{-\sqrt{\delta}y} + \frac{\xi}{\delta}$) with $g_1 g_2 = 0$.

4.2 $\phi(x,y) \neq -1$

Eliminating ϕ_4 from the equations (4.12) and using relations (4.10) and (4.11) we obtain

$$A(x,y) = \tau(x) + \frac{\phi(x,y)}{1+\phi(x,y)} g^{'}(y), \quad B(x,y) = \eta(y) + \phi(x,y)\dot{f}(x), \tag{4.18}$$
$$1 + \phi(x,y) = \frac{\epsilon_1\sqrt{c_1 - 2kf(x)}}{\epsilon_2\sqrt{c_2 + 2kg(y)}}\sqrt{\frac{g^{'}(y)^2}{\dot{f}(x)^2}},$$

$$V_1(x) = \epsilon_1\sqrt{\frac{c_1 - 2kf(x)}{\dot{f}(x)^2}}, \quad W_1(y) = \epsilon_2\sqrt{\frac{c_2 + 2kg(y)}{g^{'}(y)^2}} \tag{4.19}$$
$$V_2(x) = \frac{2i}{\hbar}\tau(x)V_1(x), \quad W_2(y) = \frac{2i}{\hbar}\eta(y)W_1(y).$$

The functions $\tau(x)$ and $\eta(y)$ are arbitrary and can be modified by gauge transformations (for instance they can be set equal to zero). The entries c_1, c_2, k, ϵ_1 and ϵ_2 are constants with $\epsilon_1^2 = \epsilon_2^2 = 1$.

The conditions (4.18) on A, B and ϕ are necessary for separation of variables in eq. (4.9). There is a further necessary condition that together with (4.18) is sufficient, namely

$$W_1(y)\phi_4(x,y) = X(x) + Y(y), \tag{4.20}$$

where $X(x)$ and $Y(y)$ are arbitrary functions.

In other words we must determine the conditions on the functions $f(x)$, $g(y)$, $\tau(x)$, $\eta(y)$ and the constants k, c_1, c_2, ϵ_1, ϵ_2 in such a manner that eq. (4.20) is satisfied. All of the above quantities are real and the square roots $\sqrt{c_1 - 2kf}$ and $\sqrt{c_2 + 2kg}$ must be simultaneously real, or simultaneously imaginary.

To proceed further we use (4.11), (4.18) and (4.19) to obtain

$$W_1(y)\phi_4(x,y) = R(x) + S(y) + T(x,y) \tag{4.21}$$

$$R(x) = -\frac{c_2\dot{f}^2(x) + (c_1 - 2kf(x))(2\lambda - 2\sigma_1 + \sigma_2 - 2\mu f(x) + \delta f^2(x) - \tau^2(x) + i\hbar\dot{\tau}(x))}{2\epsilon_1\sqrt{c_1 - 2kf(x)}\sqrt{\dot{f}(x)^2}} \tag{4.22}$$

$$S(y) = -\frac{c_1 g^{'2}(y) + (c_2 + 2kg(y))(2\epsilon - 2\lambda + \sigma_1 - 2\sigma_2 + 2\mu g(y) + \beta g^2(y) - \eta^2(y) + i\hbar\eta^{'}(y))}{2\epsilon_2\sqrt{c_2 + 2kg(y)}\sqrt{g^{'}(y)^2}} \tag{4.23}$$

and

$$T(x,y)=\frac{F_1(x)g(y)}{\sqrt{c_1-2kf(x)}}+\frac{f(x)G_1(y)}{\sqrt{c_2+2kg(y)}}$$
$$+i\hbar[F_2(x)g^{'}(y)\sqrt{\frac{c_2+2kg(y)}{g^{'}(y)^2}}+G_2(y)\dot{f}(x)\sqrt{\frac{c_1-2kf(x)}{\dot{f}(x)^2}}]. \tag{4.24}$$

In (4.24) we have

$$F_1(x)=\frac{\epsilon_1[-k\dot{f}^2(x)+(c_1-2kf(x))\ddot{f}(x)]}{\sqrt{\dot{f}(x)^2}}, \tag{4.25}$$
$$F_2(x)=\frac{\epsilon_2[k\dot{f}^2(x)+(c_1-2kf(x))\ddot{f}(x)]}{2(c_1-2kf(x))\dot{f}(x)},$$
$$G_1(y)=\frac{\epsilon_2[kg^{'2}(y)+(c_2+2kg(y))g^{''}(y)]}{\sqrt{g^{'}(y)^2}},$$
$$G_2(y)=\frac{\epsilon_1[kg^{'2}(y)-(c_2+2kg(y))g^{''}(y)]}{2(c_2+2kg(y))g^{'}(y)}.$$

The separability condition (4.20) is equivalent to the condition

$$\frac{\partial^2 T}{\partial x\partial y}=0. \tag{4.26}$$

We can consider the real and imaginary parts of eq. (4.26) separately. We obtain two conditions:

$$\frac{1}{\dot{f}}\Big(\frac{F_1}{\sqrt{c_1-2kf}}\Big)^{\cdot}=-\frac{1}{g^{'}}\Big(\frac{G_1}{\sqrt{c_2+2kg}}\Big)^{'}=N_1 \tag{4.27}$$

$$k\dot{F}_2\sqrt{\frac{c_1-2kf}{\dot{f}^2}}=kG_2^{'}\sqrt{\frac{c_2+2kg}{g^{'2}}}=N_2, \tag{4.28}$$

where N_1 and N_2 are constants.

More explicitly, eq. (4.27) and (4.28) can be rewritten as

$$\frac{-\epsilon_1[k^2\dot{f}^4+2k(c_1-2kf)\dot{f}^2\ddot{f}+(c_1-2kf)^2(\ddot{f}^2-\dot{f}\dddot{f})]}{(c_1-2kf)^{3/2}(\dot{f}^2)^{3/2}}=N_1 \tag{4.29a}$$

$$\frac{\epsilon_2[k^2g^{'4}-2k(c_2+2kg)g^{'2}g^{''}+(c_2+2kg)^2(g^{''2}-g^{'}g^{'''})]}{(c_2+2kg)^{3/2}(g^{'2})^{3/2}}=N_1 \tag{4.29b}$$

$$\frac{k\epsilon_2[2k^2\dot{f}^4 + k(c_1-2kf)\dot{f}^2\ddot{f} - (c_1-2kf)^2(\ddot{f}^2 - \dot{f}\dddot{f})]}{2(c_1-2kf)^{3/2}\dot{f}^3} = N_2 \tag{4.29c}$$

$$\frac{-k\epsilon_1[2k^2g'^4 - k(c_2+2kg)g'^2g'' - (c_2+2kg)^2(g''^2 - g'g''')]}{2(c_2+2kg)^{3/2}g'^3} = N_2 \tag{4.29d}$$

We see that the functions $\tau(x)$ and $\eta(y)$ do not figure in (4.29), and hence have no influence on the separation of variables. The functions $f(x)$ and $g(y)$ depend on the constants α, β, γ, δ and ξ of eq. (4.2) and on a total of four further integration constants. Our aim now is to find all values of these constants and of k, c_1 and c_2 for which eq. (4.29) are satisfied.

Let us consider the cases $k=0$ and $k \neq 0$ separately.

I) $k=0$

From (4.28) we have $N_2=0$ and (4.29c,4.29d) are satisfied identically. Eq. (4.29a,4.29b) simplify to

$$N_1 = \epsilon_1\sqrt{c_1}\frac{-\ddot{f}^2 + \dot{f}\dddot{f}}{(\dot{f}^2)^{3/2}} = \epsilon_2\sqrt{c_2}\frac{g''^2 - g'g'''}{(g'^2)^{3/2}}. \tag{4.30}$$

Using eq. (4.2) we obtain

$$3\epsilon_1 N_1\ddot{f} = 2\alpha\sqrt{c_1}\sqrt{\dot{f}^2}, \quad 3\epsilon_2 N_1 g'' = 2\alpha\sqrt{c_2}\sqrt{g'^2}. \tag{4.31}$$

Eq. (4.31) are only compatible with (4.2) if we have $\alpha = N_1 = 0$. The assumptions $c_1=0$ or $c_2=0$ lead to contradictions or $\phi(x,y)=-1$, so we are left with $\ddot{f}^2 - \dot{f}(\dddot{f}) = 0$, $g''^2 - g'(g''') = 0$ and hence, using (4.2) again we find that the only solutions for f(x) and g(y) are

$$\begin{aligned} f(x) &= f_1 e^{\sqrt{\beta}x} + f_2 e^{-\sqrt{\beta}x} - \frac{\gamma}{\beta} \\ g(y) &= g_1 e^{\sqrt{\delta}y} + g_2 e^{-\sqrt{\delta}y} - \frac{\xi}{\delta} \\ &\beta > 0, \delta > 0, f_1 f_2 = g_1 g_2 = 0, \end{aligned} \tag{4.32}$$

where f_i and g_i are constants, or one of the functions $f(x)$ or $g(y)$ may be linear, the other being as in (4.32).

II) $k \neq 0$

We shall run through different possible solutions of eq. (4.2) and determine which of them are compatible with eq. (4.29).

1) $\alpha = \beta = \delta = 0$, $\gamma\xi \neq 0$.

We obtain

$$f(x) = \frac{1}{2}\gamma x^2 \qquad g(y) = \frac{1}{2}\xi y^2 \tag{4.33}$$

Eq. (4.29) and the condition $(c_2 + 2kg(y))(c_1 - 2kf(x)) > 0$ imply

$$c_1 = c_2 = N_1 = N_2 = 0, \qquad \xi\gamma < 0 \tag{4.34}$$

2) $\alpha = 0, \qquad \beta^2 + \delta^2 \neq 0.$

The functions f(x) and g(y) will be expressed in terms of exponentials, trigonometric functions, or one of them may have the form (4.33). In none of these cases can eq. (4.29) be satisfied.

3) $\alpha \neq 0.$

In this case we are dealing with the two nonlinear equations (4.3). If the cubic polynomial on the right hand side has three distinct roots, we obtain solutions in terms of elliptic functions. Otherwise the solutions involve elementary functions.

Let us discuss the equation for f(x). We can assume with no loss of generality that we have $\alpha > 0$. Indeed, if we replace $f(x) \to -f(x)$,$\alpha \to -\alpha$, $\beta \to \beta$,$\gamma \to -\gamma$,$\sigma_1 \to \sigma_1$ in eq. (4.3) we get the same equation. Hence we can change the sign of α from negative to positive (if necessary). We write

$$\dot{f}(x)^2 = \frac{2}{3}\alpha(f(x) - f_1)(f(x) - f_2)(f(x) - f_3), \quad \alpha > 0 \tag{4.35}$$

We require f(x) to be real, hence $\dot{f}(x)^2 > 0$. If all three roots are real we order them as $f_1 \leq f_2 \leq f_3$. Otherwise we consider $f_1 \in \mathbb{R}$ and $f_2 = p + iq$, $f_3 = p - iq$, $p, q \in \mathbb{R}$, $q > 0$. The possibilities are:

$$a) \quad f_1 = f_2 = f_3 = -\frac{\beta}{2\alpha}, \qquad f(x) = -\frac{\beta}{2\alpha} + \frac{6}{\alpha(x - x_0)^2} \tag{4.36}$$

$$b) \quad f_1 = f_2 < f_3 < f, \quad f(x) = f_1 + \frac{f_3 - f_1}{\sin^2 \omega(x - x_0)}, \tag{4.37}$$

$$f_1 = \frac{-\beta + 2\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}, \quad \omega^2 = \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{4}, \quad (f_3 - f_1) = \frac{6\omega^2}{\alpha}$$

$$c) \quad f_1 \leq f \leq f_2 = f_3, \quad f(x) = f_1 + (f_3 - f_1)\tanh^2 \omega(x - x_0), \tag{4.38}$$

$$f_1 = -\frac{\beta + 2\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \quad \omega^2 = \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{4} \quad (f_3 - f_1) = \frac{6\omega^2}{\alpha}$$

$$d) \quad f_1 < f_2 = f_3 \leq f, \quad f(x) = f_1 + \frac{f_3 - f_1}{\tanh^2 \omega(x - x_0)}, \tag{4.39}$$

with f_1, f_3 and ω as in (4.38). In cases b,c and d we assume $\beta^2 - 4\alpha\gamma > 0$.

$$e) \quad f_1 \leq f \leq f_2 < f_3, f_1 < f_2 \quad f(x) = f_1 + (f_2 - f_1)sn^2(\omega(x - x_0), k), \tag{4.40}$$
$$\omega^2 = \frac{\alpha}{6}(f_3 - f_1), \quad k = \sqrt{\frac{f_2 - f_1}{f_3 - f_1}},$$

where $sn(\omega x, k)$ is a Jacobi elliptic function.

$$f) \quad f_1 < f_2 < f_3 \leq f, \quad f(x) = f_1 + \frac{f_3 - f_1}{sn^2(\omega(x - x_0), k)}, \tag{4.41}$$

with k and ω as in (4.40).

$$g) \quad f_1 \in \mathbb{R}, \quad f_{2,3} = p \pm iq, \quad q > 0, \quad f(x) = f_1 + A\frac{1 - cn(\rho x, k)}{1 + cn(\rho x, k)}, \tag{4.42}$$
$$\rho = \sqrt{\frac{2\alpha}{3}A}, \quad k^2 = \frac{A - f_1 + p}{2A}, \quad A^2 = (p - f_1)^2 + q^2$$

The solutions for $g(y)$ are similar and can be obtained from those for $f(x)$ by the substitutions

$$f(x) \to -g(y), \alpha \to -\alpha, \beta \to \delta, \gamma \to -\xi, \sigma_1 \to \sigma_2 \tag{4.43}$$

The solutions for $f(x)$ and $g(y)$ can now be substituted into eq. (4.29) in order to determine whether there exist constants c_1, c_2, k and ϵ_1, ϵ_2 for which the quantities N_1 and N_2 are indeed constant. It turns out that if $f(x)$ or $g(y)$ are given by elliptic functions, eq. (4.29) are never satisfied. However, if both polynomials in eq. (4.3) have multiple roots, eq. (4.29) can always be satisfied. We give the results for $f(x)$ and $g(y)$ in Tables 1 and 2, respectively. The calculations are quite cumbersome and were performed using Mathematica.

From the tables, we see that any $f(x)$ from Table 1 can be combined with any $g(y)$ from Table 2. The fact that N_2 must be the same in both tables provides a relationship between ϵ_1 and ϵ_2.

More specifically, we have:

$$\epsilon_1 = \epsilon_2 \quad for \quad (F_1, G_3), (F_2, G_3), (F_3, G_1), (F_3, G_2), (F_3, G_4), (F_4, G_3) \tag{4.44}$$

$$\epsilon_1 = -\epsilon_2 \quad for \quad (F_1, G_1), (F_1, G_2), (F_1, G_4), (F_2, G_1), (F_2, G_2), (F_2, G_4), (F_3, G_3), (F_4, G_1), (F_4, G_2), (F_4, G_4) \tag{4.45}$$

Similar tables are easily obtained for $\alpha < 0$ and we shall not present them here. We see that for $\alpha \neq 0$ we must have $k \neq 0$ ($k = 0$ would imply $c_1 = c_2 = 0$). Otherwise, k remains arbitrary, as do $\alpha, ..., \xi$ in eq. (4.2) and (4.3). The integration constants σ_1 and σ_2 must be such that the polynomials $P_3(f)$ and $Q_3(g)$ in eq. (4.3) have multiple roots. The constants c_1 and c_2 are completely determined.

Let us sum up our results as a theorem.

Theorem 1. *Separation of variables in eq. (4.6) for $\ddot{f}^2 + g''^2 \neq 0$ occurs if and only if we have one of the following*

1. $\alpha = \delta = \xi = 0$.
   In this case we have $g(y) = g_0 y + g_1$ and $f(x)$ is any solution of eq. (4.2) with $\alpha = 0$.
   We can put $\phi = -1$, $A = g'$ and $B = -\dot{f}$.
2. $\alpha = 0$, $\delta > 0$,
   $g(y) = g_1 e^{\sqrt{\delta}y} + g_2 e^{-\sqrt{\delta}y} - \frac{\xi}{\delta}$, $\quad g_1 g_2 = 0$
   and $f(x)$ is any solution of eq. (4.2) with $\alpha = 0$.
   We can put $\phi = -1$, $A = g'$ and $B = -\dot{f}$.
3. $\alpha = \beta = \delta = 0$, $\xi\gamma < 0$, $c_1 = c_2 = 0$, $k \neq 0$, $\phi = \sqrt{-\frac{\xi}{\gamma}}$.
   In this case we have $f(x) = \frac{1}{2}\gamma x^2$, $\quad g(y) = \frac{1}{2}\xi y^2$
   and we can put $A = \frac{\phi}{1+\phi} g'$ and $B = \phi \dot{f}$.
4. $\alpha \neq 0$, $k \neq 0$, *$f(x)$ and $g(y)$ are solutions of eq. (4.2) as listed in Table 1 and Table 2, respectively, and ϕ, A and B as in eq. (4.18) where we can put $\tau = \eta = 0$. The values of the constants c_1 and c_2 are listed in the tables and ϵ_1 and ϵ_2 are related as in eq. (4.44) or (4.45).*

The formulas for the vector potential (A, B) can be modified by putting $A \to A + \tau(x)$, $B \to B + \eta(y)$ without any effect on solutions.

5 Example of solving the Schrödinger equation by quasiseparation of variables

In order to show how quasiseparation of variables allows us to solve the Schrödinger equation, let us consider the simplest case, namely (4.33). We have $f(x) = \frac{\gamma x^2}{2}$, $g(y) = \frac{\xi y^2}{2}$ and hence

$$\Omega = \Omega_0 = \gamma + \xi, W = \frac{1}{2}(\xi - \gamma + \mu)(\gamma x^2 - \xi y^2), \gamma\xi < 0 \tag{5.1}$$

Changing the notation of the constants, we put

$$W = \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2), \;\; \Omega = \Omega_0, \;\; \omega_1 \neq \omega_2, \tag{5.2}$$

In eq. (4.18) we choose $\tau(x) = \eta(y) = 0$ and obtain

$$A = \frac{\omega_2 \Omega_0}{\omega_1 + \omega_2} y, \;\; B = -\frac{\omega_1 \Omega_0}{\omega_1 + \omega_2} x, \;\; \phi(x,y) = \frac{\omega_2 - \omega_1}{\omega_1}. \tag{5.3}$$

In terms of the constant magnetic field Ω_0 and the frequencies ω_1 and ω_2 the Hamiltonian H and integral X reduce to

$$H = -\frac{\hbar^2}{2}(\partial_x^2 + \partial_y^2) - i\hbar \frac{\Omega_0}{\omega_1 + \omega_2}(\omega_2 y \partial_x - \omega_1 x \partial_y) + \frac{1}{2}\frac{(\omega_1+\omega_2)^2 + \Omega_0^2}{(\omega_1+\omega_2)^2}(\omega_1^2 x^2 + \omega_2^2 y^2), \tag{5.4}$$

$$X = -\frac{\hbar^2}{2}\partial_x^2 - i\hbar \frac{\Omega_0 \omega_1}{\omega_1^2 - \omega_2^2}(\omega_2 y \partial_x - \omega_1 x \partial_y) + \frac{1}{2}\omega_1^2 \frac{(\omega_1+\omega_2)^2 + \Omega_0^2}{(\omega_1+\omega_2)^2} x^2. \tag{5.5}$$

The linear combination (4.6) of H and X that allows the separation of variables is in this case

$$\{-\frac{\hbar^2}{2}[\frac{\omega_2}{\omega_1}\partial_x^2 + \partial_y^2] + \frac{1}{2(\omega_1+\omega_2)^2}[(\omega_1+\omega_2)^2 + \Omega_0^2](\omega_1\omega_2 x^2 + \omega_2^2 y^2)\}\psi = (E - \frac{\omega_1 - \omega_2}{\omega_1}\lambda)\psi. \tag{5.6}$$

Putting

$$\psi(x,y) = v(x) w(y), \tag{5.7}$$

as in eq. (4.7) we obtain

$$\begin{gathered} -\frac{\hbar^2}{2}\frac{\omega_2}{\omega_1} v_{xx} + \frac{1}{2}\frac{\omega_1\omega_2}{(\omega_1+\omega_2)^2}[(\omega_1+\omega_2)^2 + \Omega_0^2]x^2 v + \frac{\omega_1 - \omega_2}{\omega_1}\lambda v = k_0 v, \\ -\frac{\hbar^2}{2} w_{yy} + \frac{\omega_2^2}{2(\omega_1+\omega_2)^2}[(\omega_1+\omega_2)^2 + \Omega_0^2] y^2 w - E w = -k_0 w, \end{gathered} \tag{5.8}$$

where k_0 is a separation constant and (E,λ) are fixed constants. The solutions of eq. (5.8) are of course well known and the regular, square integrable solutions are given in terms of Hermite polynomials H_n as

$$v(x) = e^{-\frac{\tau_1^2 x^2}{2}} H_{n_1}(\tau_1 x), \;\; w(y) = e^{-\frac{\tau_2^2 y^2}{2}} H_{n_2}(\tau_2 y). \tag{5.9}$$

The constants satisfy

$$\tau_a = \sqrt{\frac{\omega_a}{\hbar(\omega_1+\omega_2)}}[(\omega_1+\omega_2)^2 + \Omega_0^2]^{1/4}, \;\; a = 1,2, \tag{5.10}$$

$$
\begin{aligned}
k_0\omega_1 - (\omega_1 - \omega_2)\lambda &= \frac{\hbar\omega_1\omega_2}{2(\omega_1+\omega_2)}\sqrt{(\omega_1+\omega_2)^2+\Omega_0^2}(2n_1+1), \\
E - k_0 &= \frac{\hbar\omega_2}{2(\omega_1+\omega_2)}\sqrt{(\omega_1+\omega_2)^2+\Omega_0^2}(2n_2+1),
\end{aligned}
\tag{5.11}
$$

n_i=0,1,2,...

Eliminating k_0 we have

$$
\begin{aligned}
E\omega_1 + \lambda(\omega_2 - \omega_1) &= \frac{\hbar\omega_1\omega_2}{\omega_1+\omega_2}\sqrt{(\omega_1+\omega_2)^2+\Omega_0^2}(n+1), \\
n &= n_1 + n_2.
\end{aligned}
\tag{5.12}
$$

The solution ψ(x,y) thus depends on two nonnegative integers n_1 and n_2, whereas the constants E and λ depend only on n.

Let us now fix E and λ, i.e. fix $n = n_1 + n_2$ and write the solution of the Schrödinger equation with Hamiltonian (5.4) as a superposition of solutions of eq. (5.6):

$$
\psi_{E,\lambda}(x,y) = e^{-\frac{1}{2}(\tau_1^2 x^2 + \tau_2^2 y^2)} \sum_{n_1=0}^{n} A_{n_1,n-n_1} H_{n_1}(\tau_1 x) H_{n-n_1}(\tau_2 y),
\tag{5.13}
$$

where the constants $A_{n_1,n-n_1}$ are to be determined. We substitute (5.13) into the Schrödinger equation, use the recursion relations for the Hermite polynomials and obtain a linear homogeneous equation for the constants $A_{n_1,n-n_1}$. This equation is best written in matrix form involving a tridiagonal matrix

$$
M|A >= |0 >,
\tag{5.14}
$$

i.e.

$$
\begin{pmatrix}
\alpha_{11} & -S & 0 & & \\
nS & \alpha_{22} & -2S & \ddots & \\
0 & \ddots & \ddots & \ddots & 0 \\
 & \ddots & 2S & \alpha_{nn} & -nS \\
 & & 0 & S & \alpha_{n+1n+1}
\end{pmatrix}
\begin{pmatrix}
A_{0n} \\ A_{1n-1} \\ \vdots \\ A_{n-11} \\ A_{n0}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ \vdots \\ \vdots \\ \vdots \\ 0
\end{pmatrix}
\tag{5.15}
$$

$$
\begin{gathered}
\alpha_{11} = R(\omega_1 + (2n+1)\omega_2) - E, \;\; \alpha_{22} = R(3\omega_1 + (2n-1)\omega_2) - E, \\
\alpha_{nn} = R((2n-1)\omega_1 + 3\omega_2) - E, \;\; \alpha_{n+1n+1} = R((2n+1)\omega_1 + \omega_2) - E, \\
R = \frac{\hbar\sqrt{(\omega_1+\omega_2)^2+\Omega_0^2}}{2(\omega_1+\omega_2)}, \;\; S = \frac{i\hbar\Omega_0}{\omega_1+\omega_2}\sqrt{\omega_1\omega_2}.
\end{gathered}
\tag{5.16}
$$

The energy is obtained from an algebraic equation of order n+1, namely

$$detM = 0. \tag{5.17}$$

The constant λ, for E and n given, is obtained from eq. (5.12). The wave function for E and λ fixed has the form (5.13) with coefficients $A_{n_1 n-n_1}$ obtained by solving the system (5.15).

Thus, once eq. (5.6) is solved by separation of variables, the solution of the Schrödinger equation reduces to linear algebra. The energy operator is reduced to block diagonal form, with each block of finite dimension, namely n+1. In this scheme the problem is "exactly solvable".

For low values of n this can be done explicitly. For the ground state we have:

$$\begin{aligned} n=0 \quad E &= \frac{\hbar}{2}\sqrt{(\omega_1+\omega_2)^2+\Omega_0^2}, \;\; \lambda = \frac{\hbar\omega_1}{2(\omega_1+\omega_2)}\sqrt{(\omega_1+\omega_2)^2+\Omega_0^2}, \\ \psi_{00} &= A_{00}e^{-\frac{\tau_1^2x^2+\tau_2^2y^2}{2}}. \end{aligned} \tag{5.18}$$

The first two excited states satisfy

$$\begin{aligned} n=1 \quad E_\pm &= \frac{\hbar}{2}[2\sqrt{(\omega_1+\omega_2)^2+\Omega_0^2} \pm \sqrt{(\omega_1-\omega_2)^2+\Omega_0^2}], \\ \lambda_\pm &= \frac{\hbar}{2(\omega_2^2-\omega_1^2)}[-2\omega_1^2\sqrt{(\omega_1+\omega_2)^2+\Omega_0^2} \\ &\qquad \mp(\omega_1+\omega_2)\sqrt{(\omega_1-\omega_2)^2+\Omega_0^2}], \\ \psi_\pm &= e^{-\frac{\tau_1^2x^2+\tau_2^2y^2}{2}}[A_{01}^\pm H_1(\tau_2 y) + A_{10}^\pm H_1(\tau_1 x)], \end{aligned} \tag{5.19}$$

with

$$A_{10}^\pm = -\frac{i}{2\Omega_0}[\sqrt{(\omega_1+\omega_2)^2+\Omega_0^2}(\omega_1+3\omega_2) \pm \sqrt{(\omega_1-\omega_2)^2+\Omega_0^2}(\omega_1+\omega_2)]A_{01}^\pm. \tag{5.20}$$

For n=2 and n=3 we must solve a cubic and quartic equation, respectively.

6 Conclusions

We have shown that in the presence of a magnetic field $\Omega \neq 0$ the existence of a first order integral of motion leads to the separation of variables in the Schrödinger equation, once we make the proper choice of coordinates and gauge.

The existence of a second order integral does no longer imply the separation of variables. In many cases, identified in Theorem 1 of Section 4, we can "‘quasiseparate" variables,

that is separate variables in one equation that is an appropriate linear combination of the two equations, $H\psi = E\psi$ and $X\psi = \lambda\psi$. The solutions of the Schrödinger equation are then linear combinations of solutions of this equation, namely (4.6).

In Section 5 we analysed one of the separable cases in detail, namely that of a constant magnetic field Ω_0 and an anisotropic harmonic oscillator effective potential W (see eq. (5.2)).

Thus quasiseparation of variables is reminescent of the Dirac equation [12]. The same method can be used for all cases identified in Section 4.

The question that remains is the following: What can we do in those integrable cases when neither separation of variables, nor quasiseparation of variables occurs? The same questions arise in the case of polar integrability where the magnetic fields and effective potentials are known [2, 16], but the Schrödinger equation remains to be solved. Some examples of parabolic and elliptic integrability are known [17]; the solutions of the Schrödinger equation remain to be studied.

Finally, a few words about superintegrability [4, 5, 6, 9, 10, 11, 12, 14, 15, 16, 18, 19, 20, 21]. In two dimensions, superintegrability means that two independent operators, X_1 and X_2, commuting with H (but not with each other) exist. The only known case with a nonzero magnetic field that exists, is that of a constant magnetic field Ω_0 and a zero effective scalar potential W=0. In this case there are three first order integrals and they generate a four-dimensional Lie algebra, isomorphic to a central extension of the Euclidean Lie algebra e(2) [2].

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Table 1

Functions f(x) leading to quasiseparation of variables. We have

$$\omega_1^2 = \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{4}, \beta^2 - 4\alpha\gamma > 0, \alpha > 0, f_3 - f_1 = \frac{6\omega_1^2}{\alpha}, N_1 = \frac{8\epsilon_2 N_2}{k\epsilon_1}$$

No.	$f(x)$	c_1	N_2	Comment
F_1	$\frac{6}{\alpha(x-x_0)^2} - \frac{\beta}{2\alpha}$	$2kf_1$	$\epsilon_2 \frac{k}{2}\sqrt{\frac{-k\alpha}{3}}$	$f_1 = f_2 = f_3 \leq f(x)$ $f_1 = -\frac{\beta}{2\alpha}$
F_2	$f_1 + \frac{f_3 - f_1}{\sin^2 \omega_1(x-x_0)}$	$2kf_3$	$\epsilon_2 \frac{k}{2}\sqrt{\frac{-k\alpha}{3}}$	$f_1 = f_2 < f_3 \leq f(x)$ $f_1 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$
F_3	$f_1 + (f_3 - f_1)\tanh^2 \omega_1(x - x_0)$	$2kf_1$	$-\epsilon_2 \frac{k}{2}\sqrt{\frac{-k\alpha}{3}}$	$f_1 \leq f(x) \leq f_2 = f_3$ $f_3 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$
F_4	$f_1 + \frac{f_3 - f_1}{\tanh^2 \omega_1(x-x_0)}$	$2kf_1$	$\epsilon_2 \frac{k}{2}\sqrt{\frac{-k\alpha}{3}}$	$f_1 < f_2 = f_3 \leq f(x)$ $f_3 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$

Table 2

Functions g(y) leading to quasiseparation of variables. We have

$$\omega_2^2 = \frac{\sqrt{\delta^2 + 4\alpha\xi}}{4}, \delta^2 + 4\alpha\xi > 0, \alpha > 0, g_3 - g_1 = \frac{6\omega_2^2}{\alpha}, N_1 = \frac{8\epsilon_2 N_2}{k\epsilon_1}$$

No.	$g(y)$	c_2	N_2	Comment
G_1	$\frac{\delta}{2\alpha} - \frac{6}{\alpha(y-y_0)^2}$	$-2kg_3$	$-\epsilon_1 \frac{k}{2}\sqrt{\frac{-k\alpha}{3}}$	$g(y) \leq g_1 = g_2 = g_3$ $g_3 = \frac{\delta}{2\alpha}$
G_2	$g_3 - \frac{g_3 - g_1}{\tanh^2 \omega_2(y-y_0)}$	$-2kg_3$	$-\epsilon_1 \frac{k}{2}\sqrt{\frac{-k\alpha}{3}}$	$g(y) \leq g_1 = g_2 < g_3$ $g_1 = \frac{\delta - \sqrt{\delta^2 + 4\alpha\xi}}{2\alpha}$
G_3	$g_3 - (g_3 - g_1)\tanh^2 \omega_2(y - y_0)$	$-2kg_3$	$\epsilon_1 \frac{k}{2}\sqrt{\frac{-k\alpha}{3}}$	$g_1 = g_2 \leq g(y) \leq g_3$ $g_1 = \frac{\delta - \sqrt{\delta^2 + 4\alpha\xi}}{2\alpha}$
G_4	$g_3 - \frac{g_3 - g_1}{\sin^2 \omega_2(y-y_0)}$	$-2kg_1$	$-\epsilon_1 \frac{k}{2}\sqrt{\frac{-k\alpha}{3}}$	$g(y) \leq g_1 < g_2 = g_3$ $g_3 = \frac{\delta + \sqrt{\delta^2 + 4\alpha\xi}}{2\alpha}$

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